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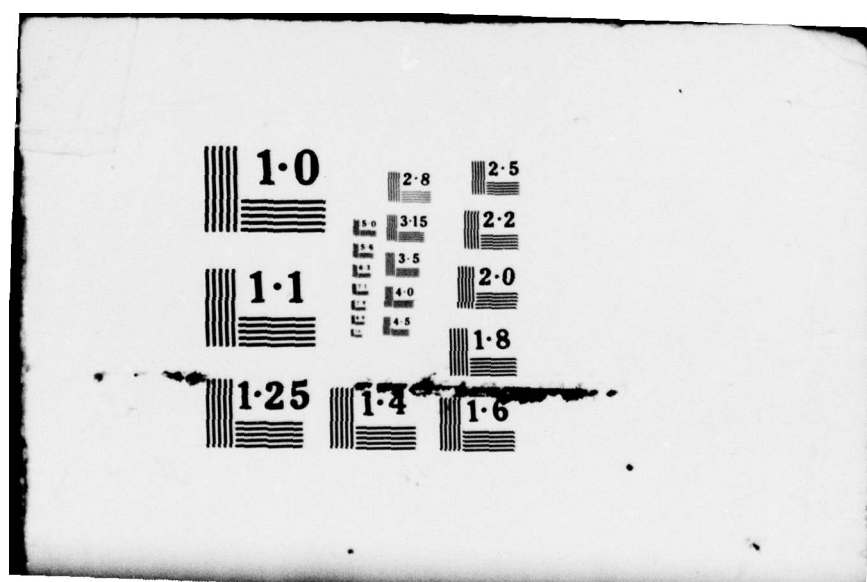
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An approximation technique for small  
noise open loop control problems

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20. Abstract continued.

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## Accompanying Note

### An approximation technique for small noise open loop control problems

This paper is concerned with the development of an approximation technique for the solution of a class of fixed stopping time small noise open loop control problems. These problems arise by adding an additive white noise term with a small coefficient  $(2\epsilon)^{1/2}I$  to the system equations in the deterministic control problem.

An approximation scheme is developed that has the advantage that one finds approximately optimal controls simultaneously for all sufficiently small  $\epsilon$ . The scheme requires the solution of a generalized linear regulator problem which is solvable easily numerically. The numerical method is given and an example illustrating the efficiency of the method is also presented.

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## 1. Introduction

This paper is concerned with the development of an approximation technique for the solution of a class of fixed stopping time small noise open loop control problems. These problems arise by adding an additive white noise term with a small coefficient  $(2\varepsilon)^{1/2}I$  to the system equations in the deterministic control problem.

In earlier work [4] we derived expansions of class  $C^\infty$  in  $\varepsilon$  of the optimal open loop cost and control for a very special class of problems in which each open loop generated a nondegenerate Gaussian process. This property allowed the conversion of the stochastic control problem into an equivalent deterministic control problem. Under less restrictive assumptions in [ ] we were able to derive a truncated expansion of the optimal cost, but were unable to theoretically establish an expansion of the optimal cost.

Motivated by these previous results, we consider more general open loop control problems in which each open loop control does not necessarily generate a Gaussian process and attempt to find "best" controls of the form  $U^0 + \varepsilon V$ . Here the function  $U^0$  denotes the optimal open loop deterministic control. This approximation scheme has the advantage that one finds approximately optimal controls simultaneously for all sufficiently small  $\varepsilon$ . This scheme leads to the selection of a control  $U^0 + \varepsilon V$  which performs better (or at least as well) than  $U^0$  in the  $\varepsilon$  problem for all sufficiently small  $\varepsilon$ . The approxi-

mation technique for the calculation of  $V$  leads to a generalized linear regulator problem which can be solved easily numerically. This scheme is superior to and does not agree with the standard secondary extremal problem as is shown in §4.

Other work on small noise problems includes the completely observable work of Fleming [1]. Other approaches to open loop control problems include Mortensen [5] and VanSlyke and Wets [6].

2. The problem. Suppose that the state  $\xi(t)$  evolves according to the stochastic differential equations

$$(1) \quad d\xi = f(t, \xi(t), U(t))dt + (2\epsilon)^{\frac{1}{2}} Idw(t)$$

where  $w$  is  $n$  dimensional Brownian motion, and with initial condition  $\xi(s_0) = x_0$ , a constant in  $R^n$ . In (1)  $U$  is a control with values in the control set  $K = R^k$ . We seek to minimize

$$(2) \quad J(U) = E\left\{\int_{s_0}^T L(t, \xi(t), U(t))dt \mid \xi(s_0) = x_0\right\}$$

over the class of open loop controls  $\mathcal{U}$ . An open loop control  $U \in \mathcal{U}$  is a Borel measurable function on  $[s_0, T]$  with values in  $K$ .

Let  $Q = [s_0, T] \times R^n$ . Throughout we assume the following:

(i) The initial point  $(s_0, x_0)$  is a fixed constant in  $R^{n+1}$ , and is known to the controller. There exists a unique optimal open loop control  $U^0$  for the deterministic control



problem (1), (2) with  $\epsilon = 0$ .

(ii)  $f(t, x, u) = A(t, x) + B(t)u$  with  $A, B$  smooth functions.

(iii)  $L$  is a smooth function and there exists  $C_0 > 0$  such that  $v^T L_{uu}(s, x, u)v \geq C|v|^2$  for all  $(s, x, u)$ .

Concerning (ii), see the remarks in §4.

The determination of the optimal control  $U^\epsilon$  for the  $\epsilon$  problem, even numerically, is impossible in general and one seeks approximations to  $U^\epsilon$ . We propose here such a scheme. Let  $U^0$  denote the optimal deterministic open loop control corresponding to starting at  $(s_0, x_0)$ . We seek a "best" approximation scheme of the form  $V^\epsilon = U^0 + \epsilon V$ .

Let  $J^\epsilon$  denote the cost function in (2) when  $\epsilon = \bar{\epsilon}$  is used in (1). Then we have the following result whose proof is contained in [2] and follows the method of §4 in [3].

Theorem 1. For each Hölder continuous function  $V$ ,

$$J^\epsilon(V^\epsilon) = J^0(U^0) + \epsilon\chi + \epsilon^2\Gamma(V) + o(\epsilon^2)$$

where  $\chi$  is independent of  $V$  and  $\Gamma(V)$  is given by

$$(3) \quad \Gamma(V) = \int_{s_0}^T [\phi_x(t, \xi^0(t), V)B(t)V(t) + \Delta_x \phi(t, \xi^0(t), V) + \frac{1}{2} V^T(t) L_{uu}(t, \xi^0(t), U^0(t)) V(t)] dt.$$

Here  $\xi^0(t)$  is the optimal trajectory for the open loop deterministic control problem with initial condition  $\xi(s_0) = x_0$

and  $\phi(s, x, V)$  satisfies

$$(4) \quad \begin{aligned} & \phi_s(s, x, V) + \phi_x(s, x, V)f(s, x, U^0(s)) + \Delta_x \psi^0(s, x) \\ & + \{L_u(s, x, U^0(s)) + \psi_x^0(s, x)B(t)\}V(s) = 0 \end{aligned}$$

on  $[s_0, T] \times R^n$  with terminal condition  $\phi(T, x) = 0$ . The function  $\psi^0(t, x)$  satisfies

$$(5) \quad \Delta_x \psi^0 + \psi_x^0 f(t, x, U^0(t)) + \psi_t^0 + L(t, x, U^0(t)) = 0$$

with terminal condition  $\psi^0(T, x) = 0$ .

Remark.  $\psi^0(t, x)$  is the cost of starting at  $(t, x)$ ,  $t \geq s_0$ , and using the open loop control  $U^0$  corresponding to the initial point  $(s_0, x_0)$ . Note that the notational dependence of  $\phi$  on  $V$  only indicates that for a fixed function  $V$ ,  $\phi$  satisfies a linear partial differential equation depending upon  $V$ .

Since  $\chi$  is independent of the choice of the Hölder continuous function  $V$ , let us attempt to choose  $V$  so as to minimize the quantity  $\Gamma(V)$ . This will be considered the "best" approximate control.

### 3. Solution of the $\Gamma(V)$ control problem.

The minimization of  $\Gamma(V)$  can be formulated as a deterministic control problem, in fact, of a generalized linear regulator type. Below, in Corollary 1 we prescribe an explicit scheme for the calculation of the minimizing  $V$ .

Define  $g_i(t) = \phi_{x_i}(t, \xi^0(t), V)$ ,  $h_{ij}(t) = \phi_{x_i x_j}(t, \xi^0(t), V)$ ,  
 and let  $g(t) = (g_1(t), \dots, g_n(t))'$ ,  $h(t) = (h_{11}(t), \dots, h_{1n}(t), \dots, h_{nn}(t))$   
 Since  $\phi_{x_i}(t, x, V)$  satisfies

$$\begin{aligned} & (\phi_{x_i})_t(t, x, V) + \phi_{xx_i}(t, x, V)f(t, x, U^0(t)) + \phi_x(t, x, V)f_{x_i}(t, x, U^0(t)) \\ (6) \quad & + (\Delta_x \psi^0)_{x_i}(t, x) + \{L_{ux_i}(t, x, U^0(t)) + \psi_{xx_i}^0(t, x)B(t)\}V(t) = 0 \end{aligned}$$

with  $\phi_{x_i}(T, x, V) = 0$ , then  $g_i(t)$  satisfies

$$\begin{aligned} \frac{dg_i(t)}{dt} &= f'_{x_i}(t, \xi^0(t), U^0(t))g(t) + (\Delta_x \psi^0)_{x_i}(t, \xi^0(t)) \\ (7) \quad & + \{L_{ux_i}(t, \xi^0(t), U^0(t)) + \psi_{xx_i}^0(t, \xi^0(t))B(t)\}V(t) \end{aligned}$$

with  $g_i(T) = 0$ ,  $i = 1, \dots, n$ . Similarly  $\phi_{x_i x_j}(t, x, V)$  satisfies

$$\begin{aligned} & (\phi_{x_i x_j})_t(t, x, V) + \phi_{xx_i x_j}(t, x, V)f(t, x, U^0(t)) + \phi_x(t, x, V)f_{x_i x_j}(t, x, U^0(t)) \\ (8) \quad & + \phi_{xx_i}(t, x, V)f_{x_j}(t, x, U^0(t)) + \phi_{xx_j}(t, x, V)f_{x_i}(t, x, U^0(t)) \\ & + (\Delta_x \psi^0)_{x_i x_j}(t, x) + \{L_{ux_i x_j}(t, x, U^0(t)) + \psi_{xx_i x_j}^0(t, x)B(t)\}V(t) = 0 \end{aligned}$$

with boundary condition  $\phi_{x_i x_j}(T, x, V) = 0$ , hence

$$\begin{aligned}
\frac{dh_{ij}(t)}{dt} &= \sum_{k=1}^n h_{kj}(t) f_{x_i}(t, \xi^0(t), U^0(t)) + \sum_{k=1}^n h_{ki} f_{x_j}(t, \xi^0(t), U^0(t)) \\
(9) \quad &+ g'(t) f_{x_i x_j}(t, \xi^0(t), U^0(t)) + (\Delta_x \psi^0)_{x_i x_j}(t, \xi^0(t)) \\
&+ \{L_{ux_i x_j}(t, \xi^0(t), U^0(t)) + \psi_{xx_i x_j}^0(t, \xi^0(t)) B(t)\} V(t)
\end{aligned}$$

with final condition  $h_{ij}(T) = 0$ . The cost function becomes

$$\begin{aligned}
(10) \quad J_4(V) &= \int_{s_0}^T \sum_{i=1}^n h_{ii}(t) + g'(t) B(t) V(t) + \\
&\quad \frac{1}{2} V'(t) L_{uu}(t, \xi^0(t), U^0(t)) V(t) dt.
\end{aligned}$$

Thus we now have a deterministic control problem with state equations (7), (9) with control function  $V$  and cost function (10). Time now runs backwards, that is, we prescribe  $h$  and  $g$  at the final time  $T$ , but the functions  $g$  and  $h$  are unspecified at time  $s_0$ . The quantities  $\psi_{x_i x_i}^0(t, \xi^0(t))$  and  $\psi_{x_k x_i x_j}^0(t, \xi^0(t))$  can be found easily using the method of characteristics once  $U^0(t)$  is known. One simply repeats the procedure on  $\psi^0$  used in deriving equations (7) and (9).

We now formulate a generalized linear regulator problem for

$$z(t) \stackrel{\text{defn}}{=} (g_1, \dots, g_n, h_{11}, \dots, h_{1n}, \dots, h_{n1}, \dots, h_{nn})(T-t).$$

(Of course,  $h_{ij} = h_{ji}$  so in actual numerical computation some of the terms may be eliminated.) The equations for  $z$  can be



written in the form

$$(11) \quad \frac{dz}{dt} = D_1 z + D_2 w + E,$$

$z(0) = 0$ , with cost function,

$$(12) \quad J_5(w) = \int_0^{T-s_0} K^T z + z^T R w + \frac{1}{2} w^T Q w dt$$

for appropriate matrices  $D_1, D_2, R, Q$  ( $R, Q$  symmetric) and vectors  $E, K$ , and control function  $w(t) = V(T-t)$ .

This problem can be solved using dynamic programming. Let  $\phi(t, z)$  be the optimal cost corresponding to the control problem (11), (12) but with initial condition  $z(t) = z$  instead of  $z(0) = 0$ . Then  $\phi$  satisfies the Hamilton-Jacobi equation

$$(13) \quad \begin{aligned} \phi_t + \phi_z D_1 z + \phi_z E + K^T z \\ + \min_w [(\phi_z D_2 + z^T R) w + \frac{1}{2} w^T Q w] = 0. \end{aligned}$$

The minimum in (13) is obtained when

$$(14) \quad w = -Q^{-1} [\phi_z D_2 + z^T R]^T$$

hence (13) can be written as

$$(15) \quad \begin{aligned} \phi_t + \phi_z (D_1 z + E) + (K^T z) \\ - \frac{1}{2} (\phi_z D_2 + z^T R) Q^{-1} (\phi_z D_2 + z^T R)^T = 0. \end{aligned}$$

This equation has the solution

$$(16) \quad \phi(t, z) = \frac{1}{2} z^T P(t) z + r^T(t) z + q(t)$$



where

$$\begin{aligned} \frac{1}{2} P' + P^T D_1 - \frac{1}{2} R Q^{-1} R^T - \frac{1}{2} P^T D_2 Q^{-1} D_2^T P \\ - R Q^{-1} P = 0, \end{aligned}$$

$$\begin{aligned} (r')^T + r^T D_1 + K^T \\ - r^T D_2 Q^{-1} D_2^T P - r^T D_2^T Q^{-1} R^T = 0, \end{aligned}$$

$$q' + r^T E - \frac{1}{2} r^T D_2 Q^{-1} D_2^T r = 0,$$

and with initial conditions

$$q(0) = 0, \quad r^T(0) = 0, \quad P(0) = 0.$$

Using (14) one obtains that the optimal feedback control is

$$(17) \quad \tilde{w}(t, z) = -Q^{-1} [D_2^T r + D_2^T P^T z + R^T z].$$

Therefore we have the following

Corollary 1. The function  $V^*(t)$  minimizing  $\Gamma(V)$  is given by

$$(18) \quad V^*(t) = \tilde{w}(T-t, z^0(T-t))$$

where  $z^0(t)$  is the solution to (11) with  $w = \tilde{w}(t, z)$  and  $z(0) = 0$ .

#### 4. Conclusions.

Example 1. Consider scalar equations

$$d\xi(t) = U(t)dt + (2\varepsilon)^{\frac{1}{2}} dw(t),$$

$\xi(0) = 0$ , and cost function

$$E \int_0^1 [(\xi(t))^2 + \xi(t))^2 + \xi(t)^2 + \frac{1}{2} U(t)^2] dt.$$

This problem is actually of the type considered in [4], but let us use the methods of the paper to determine the optimal  $V^*$ . Since  $U^0 \equiv 0$ , then  $\psi^0(t, x) = (1-t)(x^4 + 2x^3 + 2x^2)$  and the deterministic control problem for  $V$  is the following.

Minimize  $\int_0^1 h_{11}(t) + g_1(t)V(t) + V^2(t)dt$  with state equations

$$\frac{dg_1(t)}{dt} = 12(1-t) + 4(1-t)V(t), \quad g_1(1) = 0,$$

$$\frac{dh_{11}(t)}{dt} = 24(1-t) + 12(1-t)V(t), \quad h_{11}(1) = 0,$$

over the class of open loop controls  $V(t)$ . Rather than use the procedure of Section 3, we use Pontryagin's maximum principle to determine  $V$ .  $V$  is determined from the equation

$$V(t) + g_1(t) + 4p_1(t)(1-t) + 12p_2(t)(1-t) = 0$$

where  $p_1(t)$  and  $p_2(t)$  are the costate variables which satisfy

$$\frac{dp_1(t)}{dt} = -V(t), \quad p_1(0) = 0,$$

and

$$\frac{dp_2(t)}{dt} = -1, \quad p_2(0) = 0.$$

It is easily verified that  $V(t) = -3(1 - (\text{sech } 2)\cosh 2t)$  satisfies the above equations. Recall that  $V^\epsilon = U^0 + \epsilon V$  is then the best approximate control. The costs of using  $V^\epsilon$  and  $U^0$  in the  $\epsilon$ -problem for various  $\epsilon$  are listed in Table 1.

$\epsilon$	Cost Using $U^0$	Cost Using $V^\epsilon$ - Cost Using $U^0$
0	0	0
.04	.0864	-.00232
.08	.1856	-.00714
.12	.2976	-.011946
.16	.4224	-.01375
.20	.56	-.00952
.24	.7104	.00388
.40	1.44	.21644
.80	4.16	3.20844
1.00	6	7.08288

Table 1

For  $\epsilon = +.12$  the use of  $V^\epsilon$  realizes an approximately 4% decrease in cost over the cost of using  $U^0$ . However, note that as  $\epsilon$  increases, the use of  $V^\epsilon$  realizes more cost than using  $U^0$  in the  $\epsilon$ -problem.

Remark. The "best" control approximation technique is admittedly complex. In partial justification for such a complex scheme, let us show that a less complicated scheme - an accessory stochastic control problem similar to that for the deterministic control problem by [ 7 ] yields a trivial and unusable solution. Consider linear state equations of the form

$$d\xi(t) = A(t)\xi(t) + B(t)U(t)dt + (2\varepsilon)^{1/2}Idw, \quad \xi(s_0) = x_0,$$

$U(t) \in R^k$ , with cost function  $L$ . Define  $x(t) = \xi(t) - \xi^0(t)$ ,  $V(t) = U(t) - U^0(t)$ , then  $x(t)$  satisfies the equation  $dx = A(t)x + B(t)V(t)dt + (2\varepsilon)^{1/2}dw$ ,  $x(s_0) = 0$ , with cost function

$$E \int_{s_0}^T L(t, \xi(t), U(t)) - L(t, \xi^0(t), U^0(t)) dt. \quad \text{Since}$$

$$E \int_{s_0}^T L_x(t, \xi(t), U(t)) (\xi(t) - \xi^0(t)) + L_u(t, \xi^0(t), U^0(t)) (U(t) - U^0(t)) dt = 0,$$

then the new cost function can be approximated by

$$E \int_{s_0}^T (x(t), V(t)) \begin{pmatrix} L_{xx}^0 & L_{xu}^0 \\ L_{ux}^0 & L_{uu}^0 \end{pmatrix} (x(t), V(t))' dt$$

where the  $^0$  indicates evaluation along  $(t, \xi^0(t), U^0(t))$ . If the matrix of partial derivatives of  $L$  is positive definite, then this approximate control problem is minimized by the choice  $V(t) \equiv 0$  since  $x(s_0) = 0$ . Thus this linearization technique to compute a correction factor yields zero correction. However, Example 1 was of the above type and a correction term yielded a lower cost than using  $U^0$  for sufficiently small  $\varepsilon$ .



Remark. The approximation technique described in this chapter can also be used if  $B = B(t, x)$ . However, the equations for  $g(t)$ ,  $h(t)$  are complicated slightly by the addition of terms involving the  $x$ -partial derivatives of  $B$  evaluated along  $(t, \xi^0(t))$ .

Remark. The original work on the problem was done in an unpublished part of the author's dissertation [2]. Recently, we have discovered the convenient solution to the auxiliary minimization problem which was lacking in [2], and which make the auxiliary problem tractable for large scale systems.

#### References

1. W. Fleming, Stochastic control for small noise intensities, SIAM J. Control 9(1971), 483-517.
2. C. Holland, Small noise open loop problems, Ph.D. thesis Brown University (1972).
3. C. Holland, Small noise open loop control, SIAM J. Control 12(1974), 380-388.
4. C. Holland, Gaussian open loop control problems, SIAM J. Control 12(1975), 545-552.
5. R. Mortensen, Stochastic control with noisy observations, Internat. J. Control 4(1966), 455-464.
6. R. Van Slyke and R. Wets, Programming under uncertainty and stochastic optimal control, SIAM J. Control 4(1966), 179-193.
7. J. V. Breakwell, J. L. Speyer, and A. E. Bryson, "Optimization and control of nonlinear systems using the second variation", SIAM J. Control 1(1963), 193-223.